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Open Systems of Splitting Particles

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Systems with an infinite variety of types of splitting particles are investigated. It is shown that if there is a stationary source of particles but no sink, a steady state with finite density of each species is nevertheless possible due to the infinite number of degrees of freedom. It is demonstrated that the limiting (steady) state is independent of the initial state of the system. Typical features of the steady state, which do not depend on the particle splitting law, are shown.

KEY WORDS: Open systems with infinite variety of splitting particles; branching; renewal theory equation; steady state; particle density; Malthus parameter.

1. INTRODUCTION

There are physical systems composed of particles which can be either split or multiplied in addition to their collisions and drift in force fields. Typical systems like this are rock particles,[(1)] drops of liquid in turbulent flow,[(2)] gas bubbles in liquid, bloodstreams in a continuously variable system of branching blood vessels,[(3)] wave packets in three-wave mixing, etc.

Kolmogorov[(1)] found that the posterity of the multiple splitting of a rock particle has a logarithmically normal distribution in size (see also refs. 4 and 5.

It should be interesting to know the type of distribution of particles in the presence of an external source. A typical system like this is a rock crusher where the product is fed continuously.

An example of such an equilibrium system is possibly gas bubbles in the surface layer of the ocean.

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In a circulatory system blood is fed continuously from the heart ventricles to a network of small vessels which rearranges as those vessels are opened or closed.

We will demonstrate that in a system with an infinite variety of splitting particles in the presence of a stationary source, none of the particle species is infinitely accumulated with time, but a stationary limiting state with the finite density of each species is established even in the absence of a sink. If the particle species are numbered by a scalar parameter k (mass, size, energy, etc.) and the probability distribution of parameters for posterity of any particle depend only on the ratio of the parameters of descendents to the parameter of this particle, then for small k the particle density is close to σ/k^α in the steady state, where σ and α are determined from the source intensity and statistics of the splitting.

This article is a revision of an earlier preprint.[6]

2. EVOLUTION EQUATIONS FOR GENERATING FUNCTIONAL AND PARTICLE DENSITY

Consider a system of splitting particles which have either a scalar parameter (e.g., mass) or a vector parameter (e.g., the wave vector of wave packet). Assume that the following conditions are met:

(a) The particles split independently.

(b) A k-type particle disintegrates into n particles in a unitary splitting event with the probability $q_n(k)$; the frequency distribution function of division of a k-type particle into n $k_1,..., k_n$ type particles is $\mathscr{P}_n(k/k_1,..., k_n)$.

(c) The particle lifetimes are statistically independent; the splitting rate of a k-type particle is $a(k)$.

(d) The particles are produced from a source with the particle number distribution function $\{r_n\}$, $0 \leqslant n < \infty$, and the species frequency distribution functions $\{G_n(k_1,..., k_n)\}$, $0 \leqslant n \leqslant \infty$, with the rate b.

2.1. The state of a system with a variable number of particles can be described by the generating functional[7]

$$F(z) = \sum_{n=0}^{\infty} Q_n \int z(k_1) \cdots z(k_n)\, dp_n(k_1,..., k_n)$$

where $\{Q_n\}_{n=0}^{\infty}$ and $\{p_n(k_1,..., k_n)\}$ are the probability distributions; the distributions $p_n(k_1,..., k_n)$ are symmetrical with respect to replacement of the variables $k_1,..., k_n$.

An important characteristic of the state is the particle density

$$x(k) = \sum_{n=0}^{\infty} nQ_n \hat{\mathscr{P}}_n(k)$$

where

$$\hat{\mathscr{P}}_n(k) = \int \mathscr{P}_n(k_1, ..., k_{n-1}, k_n)\, dk_1 \cdots dk_{n-1}$$

$\mathscr{P}_n$ is the probability density of p_n. The mean number of particles $N(D)$ in the region $\mathscr{D}$ of the space of particle species is given by

$$N(\mathscr{D}) = \int_{\mathscr{D}} x(k)\, dk$$

The particle density can be easily calculated[7] using the generating functional

$$x(k) = \left.\frac{\delta F(z)}{\delta z}\right|_{z=1}$$

where the function $\delta F(z, k)/\delta z$ of the parameter k is defined by the following relation for any function h:

$$\left\langle \frac{\delta F}{\delta z}, h \right\rangle = \left.\frac{d}{dt}\right|_{t=0} F(z + th)$$

where $\langle g, h \rangle = \int g(k)\, h(k)\, dk$.

2.2. Consider first a system of splitting particles in the absence of a source. We denote the generating functional at the time t, provided that initially there is one k-type particle, by $F_k(t, z)$. Since the particle splitting is a branching Markov random process, the following relation should be satisfied for any $t, s > 0$ (for the finite-dimensional case see ref. 8):

$$F_k(s + t, z) = F_k(t, F^{s,z}(k))$$

where $F^{s,z}(k) = F_k(s, z)$ [i.e., $F^{s,z}(k)$ is $F_k(s, z)$ regarded as a function of k for fixed s, z]. From this relation it follows that

$$\frac{\partial F_k(t, z)}{\partial t} = \left\langle \frac{\delta F_k(t, z)}{\delta z}, \left.\frac{\partial F^{s,z}}{\partial s}\right|_{s=0} \right\rangle$$

At $s \to 0$ we have, by conditions (a)–(c), that

$$F_k(s, z) = z(k) + sa(k)[f_k(z) - z(k)] + O(s)$$

where

$$f_k(z) = \sum_{n=0}^{\infty} q_n(k) \int z(k_1) \cdots z(k_n) \mathscr{P}(k/k_1, ..., k_n) \, dk_1 \cdots dk_n$$

Denoting for fixed z

$$f^z(k) = f_k(z)$$

we get the equation

$$\frac{\partial F_k(t, z)}{\partial t} = \left\langle \frac{\delta F_k(t, z)}{\partial z}, a(f^z - z) \right\rangle \tag{1}$$

2.3. To obtain the equation for generating a functional system of splitting particles with a source of particles, we introduce an imaginary nonvanishing particle which produces all other particles according to the law given in condition (d). From Eq. (1) it follows that the generating functional $F_0(z, z_0, t)$ of this system satisfies the equation

$$\frac{\partial F_0}{\partial t} = \left\langle \frac{\delta F_0}{\delta z}, a(f^z - z) \right\rangle + b[g(z) z_0 - z_0] \frac{\partial F_0}{\partial z_0}$$

where

$$g(z) = \sum_{n=0}^{\infty} r_n \int \cdots \int z(k_1) \cdots z(k_n) \, G_n(k_1, ..., k_n) \, dk_1 \cdots dk_n$$

and r_n, Q_n are defined in condition (d). However, $F_0(z, z_0, t) = z_0 F(z, t)$, where $F(z, t)$ is the generating functional of state of a system of splitting particles which had no particles at $t = 0$. Therefore, $F(z, t)$ satisfies the equation

$$\frac{\partial F}{\partial t} = \left\langle \frac{\delta F}{\delta z}, a(f^z - z) \right\rangle + b(g(z) - 1) F(z) \tag{2}$$

Calculating the functional derivatives from both sides of Eq. (2) at $z = 1$, we obtain the following equation for particle density:

$$\frac{\partial x(k, t)}{\partial t} = \int W(k, k') \, x(k', t) \, dk' + \eta(k) \tag{3}$$

where

$$W(k, k') = a(k') \left[\sum_{n=0}^{\infty} q_n(k')\, n\hat{\mathscr{P}}_n(k'/k) - \delta(k-k') \right]$$

$$\hat{\mathscr{P}}_n(k'/k) = \int \mathscr{P}_n(k'/k_1,..., k_{n-1}, k)\, dk_1 \cdots dk_{n-1}$$

$$\eta(k) = b \sum_{n=0}^{\infty} \hat{G}(k)\, r_n(k)$$

$$\hat{G}_n(k) = \int \cdots \int G_n(k_1,..., k_{n-1}, k)\, dk_1 \cdots dk_{n-1}$$

[see condition (d)].

Equations for all other moments of state can be derived from (2) in the same way.

3. THE STEADY-STATE DENSITY OF PARTICLES

3.1. We now consider more mathematically and in more detail the splitting of scalar-type particles under the following assumptions:

$$\mathscr{P}_n(k/k_1,..., k_n) = R_n(k_1/k,..., k_n/k)\, k^{-n}$$

and r_n, q_n, and a are independent of k. In this case

$$W(k, k') = aQ(k/k')/k$$

where

$$dQ(\tau) = \sum_{n=1}^{\infty} nq_n \int \cdots \int dR_n(\tau_1,..., \tau_{n-1}, \tau) \tag{4}$$

If we suppose that supp $dR_n \subset [0, 1]^n$, then Eq. (3) becomes

$$\frac{\partial x}{\partial t} = aBx - ax + \eta \tag{5}$$

where the operator B is defined as

$$[Bx](k) = \int_0^1 \frac{1}{\tau} x(k/\tau)\, dQ(\tau) \tag{6}$$

If the measure dQ has a compact support contained in the interval [0, 1], then the integral in the left-hand side of Eq. (6) exists for any k, provided that the function x is continuous and x has a compact support. We define α_Q as a real root of the equation

$$\mu_\alpha = 1 \tag{7}$$

where $\mu_\alpha = \int_0^1 \tau^\alpha \, dQ(\tau)$. If this root exists, it is called the Malthus parameter. Equation (7) has not more than one real root if $dQ \neq \delta(\tau - 1)\, d\tau$. Obviously, $\mu_\alpha < \mu_\beta$ for $\alpha > \beta$.

Theorem 1. Let the measure dQ in Eq. (6) be finite, the Malthus parameter α_Q exist, and for some $\alpha > \alpha_Q$ and $j = 0, ..., [\alpha - \alpha_Q]$ the functions $(\partial^j/\partial k^j)\, k^\alpha x_0(k)$, $(\partial^j/\partial k^j)\, k^\alpha \eta$ belong to $L_p([0, \infty))$ for some p, $1 \leqslant p \leqslant \infty$, or to $C_0([0, \infty))$. Then we have:

(1) The stationary equation

$$aBx_\infty - ax_\infty + \eta = 0 \tag{8}$$

has a unique solution x_∞, such that for $j = 0, ..., [\alpha - \alpha_Q]$, the functions $(\partial^j/\partial k^j)\, k^\alpha x_\infty$ belong to $L_p([0, \infty))$ or $C_0([0, \infty))$, respectively.

(2) The solution $x(t, k)$ of initial value problem (5), $x(0) = x_0$, obeys the inequalities

$$\begin{aligned} &\left\| \frac{\partial^j}{\partial k^j} k^\alpha x(t) - \frac{\partial^j}{\partial k^j} k^\alpha x_\infty \right\|_{L_p} \\ &\quad \leqslant e^{-a(1-\mu_\alpha)t} \left(\left\| \frac{\partial^j}{\partial k^j} k^\alpha x_0 \right\|_{L_p} + \frac{1}{a(1-\mu_\alpha)} \|\eta^{\alpha_j}\|_{L_p} \right) \end{aligned} \tag{9}$$

Proof. For any $x \in L_p([0, \infty))$, denote $x^{\alpha j}(k) = (\partial^j/\partial k^j)\, k^\alpha x(k)$. If $x \in C_0^\infty([0, \infty))$, then

$$\frac{\partial^j}{\partial k^j} k^\alpha Bx = B^{\alpha j} x^{\alpha j}$$

where

$$[B^{\alpha j} x^{\alpha j}](k) = \int_0^1 \tau^{\alpha - j - 1} x^{\alpha j}(k/\tau)\, dQ(\tau)$$

From the Minkowski inequality it follows that

$$\|B^{\alpha j} x^{\alpha j}\|_{L_p} = \left\{ \int_0^\infty \left| \int_0^1 \tau^{\alpha - j - 1} x^{\alpha j}(k/\tau)\, dQ(\tau) \right|^p dk \right\}^{1/p} \leqslant \mu_{\alpha - j} \|x^{\alpha j}\|_{L_p}$$

Therefore

$$\|B^{\alpha j}\| \leqslant \mu_{\alpha - j} < \mu_Q = 1, \qquad j = 0, \ldots, [\alpha - \alpha_Q]$$

and the linear operator $(I - B^{\alpha j})^{-1}$ is continuous in $L_p([0, \infty))$. It follows that

$$x_\infty^{\alpha j} = \frac{1}{a} (I - B^{\alpha j})^{-1} \eta_\infty^{\alpha j}$$

and, therefore, $x_\infty^{\alpha j} \in L_p([0, \infty))$ for $j = 0, \ldots, [\alpha - \alpha_k]$ and Proposition 1 is valid. Since

$$\begin{aligned} x^{\alpha j}(t) &= e^{-ta(I - B^{\alpha j})} x_0^{\alpha j} + \int_0^t e^{-a(t-s)(I - B^{\alpha j})} \eta^{\alpha j} \, ds \\ &= \frac{1}{a} (I - B^{\alpha j})^{-1} \eta^{\alpha j} + e^{-at(I - B^{\alpha j})} \left[x_0^{\alpha j} + \frac{1}{a} (I - B^{\alpha j})^{-1} \eta^{\alpha j} \right] \end{aligned}$$

we have

$$\|x^{\alpha j}(t) - x_\infty^{\alpha j}\| \leqslant \|e^{-ta(I - B^{\alpha j})}\| \left(\|x_0^{\alpha j}\| + \frac{(1 - \mu_\alpha)^{-1}}{a} \|\eta^{\alpha j}\| \right)$$

Taking into account the inequality

$$\|e^{-ta(I - B^{\alpha j})}\| \leqslant e^{-ta(1 - \mu_\alpha)}$$

we get Proposition 2.

3.2. Equation (8) describing the steady state of the system in question is close to some equation of the renewal theory. Due to this relation we can state more precisely the results concerning the steady-state density of particles $x_\infty(k)$.

Theorem 2. Let $\operatorname{supp} dQ \subset [0, 1]$, the measure $dQ(e^u)$ be not arithmetic, and $\eta(k)$ be a continuous function having a compact support. Then for $k \to \infty$

$$x_\infty(k) = \frac{\sigma}{k^{\alpha_Q + 1}} [1 + O(k)] \tag{10}$$

where

$$\sigma = \frac{\int_0^\infty \eta(k) \, k^{\alpha_Q} \, dk}{a \int_0^1 |\ln \tau| \, \tau^{\alpha_Q} \, dQ(\tau)}$$

Proof. If $\eta(k)=0$ for $k \geqslant k_m$, $k_m < \infty$, and $\operatorname{supp} Q \subset [0, 1]$, then $x_\infty(k)=0$ for $k > k_m$ and Eq. (8) takes the form

$$a \int_{k/k_m}^{1} x_\infty(k/\tau)(1/\tau)\, dQ(\tau) - a x_\infty(k) + \eta(k) = 0$$

Denote

$$k = k_m e^{-t}, \qquad \tau = e^{-u}, \qquad k(t) = x_\infty(k_m e^{-t})$$
$$f(t) = \eta(k_m e^{-t}), \qquad dG(u) = e^u\, dQ\,(e^{-u})$$

Then $k(t)$ obeys the equation

$$a \int_0^t k(t-u)\, dG(u) - ak(t) + f(t) = 0$$

Under the assumption of the theorem, the following relation is valid:

$$k(t) \sim n_f e^{\beta_Q t}$$
$$n_f = \frac{\int_0^\infty f(t)\, e^{-\beta_Q t}\, dt}{a \int_0^\infty t e^{-\beta_Q t}\, dG(t)}$$

where β_Q is the unique solution of the equation

$$\int_0^\infty e^{-\beta_Q t}\, dG(t) = 1$$

(see ref. 8, Chapter VI). But this equation is equivalent to

$$\int_0^1 \tau^{\beta_Q - 1}\, dQ(\tau) = 1$$

Therefore, $\beta_Q = \alpha_Q + 1$. Finally,

$$n_f = \frac{\int_0^\infty \eta(k_m e^{-t})\, e^{-\beta_Q t}\, dt}{a \int_0^\infty t e^{-\beta_Q t} e^t\, dQ(e^{-t})} = k_m^{-\beta_Q} \frac{\int_0^\infty \eta_k k^{\beta_Q - 1}\, dk}{a \int_0^1 |\ln(\tau)|\, \tau^{\beta_Q - 1}\, dQ}$$

The theorem is proved.

3.3. In some special case we can find the value of the Malthus parameter α_Q. Suppose that

$$\operatorname{supp} dR_n(\tau_1,..., \tau_n) \subset \{(\tau_1,..., \tau_n): \tau_1 + \cdots + \tau_n = 1\}$$

i.e., the value k is conserved during the disintegration: for instance, the sum of masses of descendent particles is equal to the mass of the parent particle. Then

$$\int_0^1 \tau\, dQ(\tau) = \sum_{n=1}^{\infty} nq_n \int_0^1 \cdots \int_0^1 \tau_n\, dR_n(\tau_1,..., \tau_n)$$
$$= \sum_{n=1}^{\infty} nq_n \int_0^1 \cdots \int_0^1 \frac{\tau_1 + \cdots + \tau_n}{n}\, dR_n(\tau_1,..., \tau_n) = 1$$

and, consequently, $\alpha_Q = 1$.

3.4. Let us calculate the throughput of a rock crusher. Let

$$x_\infty^{\bar{k}}(k) \overset{\text{def}}{=} \begin{cases} x_\infty(k), & k \leqslant \bar{k} \\ 0, & k > \bar{k} \end{cases}$$

be the steady-state particle density of a subsystem of all particles with $k \leqslant \bar{k}$. Then $x_\infty^{\bar{k}}(k)$ obeys the equation

$$aBx_\infty^{\bar{k}} - ax_\infty^{\bar{k}} + \eta^{\bar{k}} = 0$$

where

$$\eta^{\bar{k}}(k) = \eta(k) + a \int_{k/k_m}^{k/\bar{k}} x_\infty(k/\tau)(1/\tau)\, dQ(\tau)$$

for $k \leqslant \bar{k}$ and $\eta^{\bar{k}}(k) = 0$ for $k > \bar{k}$. The function

$$J(\bar{k}) = \int_0^{\bar{k}} [\eta^{\bar{k}}(k) - \eta(k)]\, dk$$

can be taken as a stationary flux of the sink in a subsystem of particles with $k > \bar{k}$.

Proposition 1. If the measure dQ satisfies the conditions of Theorem 2, then at $\bar{k} \to 0$

$$J(\bar{k}) = \left[\frac{a\sigma}{\alpha_Q} + O(\bar{k})\right] \bar{k}^{-\alpha_Q} \tag{11}$$

Proof. If $I_1(\bar{k})$ and $I_2(\bar{k})$ are defined as

$$I_1(\bar{k}) = a \int_0^{\bar{k}} dk \int_{k/k_m}^{k/(kk_m)^{1/2}} x_\infty(k/\tau)(1/\tau)\, dQ(\tau)$$

$$I_2(\bar{k}) = a \int_0^{\bar{k}} dk \int_{k/(\bar{k}k_m)^{1/2}}^{k/\bar{k}} x_\infty(k/\tau)(1/\tau)\, dQ(\tau)$$

then

$$J(\bar{k}) = I_1(\bar{k}) + I_2(\bar{k})$$

Since $k/\tau \geqslant (\bar{k}k_m)^{1/2}$ for all τ belongs to the interval $[k/k_m, k/(\bar{k}k_m)^{1/2}]$, we have

$$x_\infty(k/\tau) \leqslant C[(\bar{k}k_m)^{1/2}]^{-\alpha_Q - 1}$$

for all k, τ in the integral $I_1(\bar{k})$. We therefore get the estimate

$$I_1(k) < C_1 \bar{k}^{(1-\alpha_Q)/2}$$

On the other hand,

$$x_\infty(k/\tau) = (k/\tau)^{-\alpha_Q - 1}\,[a\sigma + O(k/\tau)]$$

and $k/\tau \leqslant (\bar{k}k_m)^{1/2}$ for all τ belongs to the interval $[k/(\bar{k}, k_m)^{1/2}, k/\bar{k}]$. It follows that

$$x_\infty(k/\tau) = (k/\tau)^{-\alpha_Q - 1}\,[a\sigma + O(\bar{k})]$$

and

$$\begin{aligned} I_2(\bar{k}) &= \int_0^{\bar{k}} dk \int_{k/(\bar{k}k_m)^{1/2}}^{k/\bar{k}} [a\sigma + O(\bar{k})](k/\tau)^{-\alpha_Q - 1}\,(1/\tau)\,dQ(\tau) \\ &= [\sigma a + O(\bar{k})] \int_0^1 \tau^{\alpha_Q}\,dQ(\tau) \int_{\bar{k}\tau}^{(\bar{k}k_m)^{1/2}} k^{-\alpha_Q - 1}\,dk \\ &= [\sigma a/\alpha_Q + a(\bar{k})]\,\bar{k}^{-\alpha_Q} \end{aligned}$$

This relation completes the proof.

4. SOME EXACT SOLUTIONS OF THE STEADY-STATE PARTICLE DENSITY EQUATION

4.1. Let us consider Eq. (8) for $dQ = \sum_{j=1}^{N} a_j \tau^{\alpha_j}\,d\tau$. Denote $\beta = -\alpha - 1$. Then Eq. (7) takes the form

$$\sum_{j=1}^{N} a_j \frac{1}{\alpha_j - \beta} = 1 \tag{12}$$

and $\beta_Q = -\alpha_Q - 1$ is the largest root of this equation.

Define also some auxiliary operators M_τ, F_β

$$\begin{aligned} &[M_\tau f](k) = f(k/\tau) \\ &[F_\beta f](k) = \int_0^1 \tau^\beta M_{1/\tau} f \, d\tau = k^{\beta+1} \int_k^\infty f(\xi)\, \xi^{\beta-2}\, d\xi \end{aligned} \tag{13}$$

Then $B = \sum_{j=1}^N a_j F_{\alpha_j}$ and Eq. (8) takes the form

$$a \sum_{j=1}^N a_j F_{\alpha_j} x_\infty - a x_\infty + \eta = 0 \tag{14}$$

Lemma. If $\beta_1,..., \beta_N$ are different roots of Eq. (12), $\beta_l \neq \alpha_j$ and $j, l = 1,..., N$, so that the system of linear equations

$$\sum_{l=1}^N \frac{b_l}{\alpha_j - \beta_l} = 1, \qquad j = 1,..., N \tag{15}$$

has some solution $b_1,..., b_N$, then the operator inverse to $(I - B)$ in Eq. (14) has the form

$$(I - B)^{-1} = I + \sum_{l=1}^N b_l F_{\beta l} \tag{16}$$

Proof. If $\alpha \neq \beta$, then

$$F_\alpha F_\beta = \int_0^1 \int_0^1 \tau_1^\alpha \tau_2^\beta M_{1/\tau_1\tau_2}\, d\tau_1\, d\tau_2 = \frac{1}{\alpha - \beta}(F_\beta - F_\alpha)$$

For any $a_1,..., a_N$, $b_1,..., b_N$ it follows that

$$\begin{aligned} &\left(I - \sum_{j=1}^N a_j F_{\alpha_j}\right)\left(I + \sum_{l=1}^M b_l F_{\beta_l}\right) \\ &\quad = I + \sum_{l=1}^N b_l \left(\sum_{j=1}^N \frac{a_j}{\alpha_j - \beta_l} - 1\right) F_{\beta l} \\ &\qquad + \sum_{j=1}^N a_j \left(\sum_{l=1}^M \frac{b_l}{\alpha_j - \beta_l} - 1\right) F_{\alpha j} \end{aligned}$$

Taking into account Eqs. (12) and (15), we get (16).

4.2. We now consider some examples of the exact solutions of Eq. (14).

Example 1. Let $Q(\tau) = a_1$. Then Eqs. (12) and (15) have the solutions $\beta_1 = -a_1 - 1$, $b_1 = a_1$. Hence, in view of Eqs. (13) and (16), we have

$$x_\infty(k) = \frac{1}{a}\eta(k) + \frac{a_1}{a} k^{-a_1} \int_k^\infty \eta(\xi)\, \xi^{a_1 - 1}\, d\xi$$

and at $k \to \infty$ we have

$$x_\infty(k) \cong k^{-(\alpha_Q+1)} \int_0^\infty \eta(\xi)\, \xi^{\alpha_Q}\, d\xi$$

which coincides with Eq. (10), since

$$\int_0^1 |\ln \tau|\, \tau^{a_1-1}\, d\tau = \frac{1}{a_1^2}$$

Example 2. Let $Q(\tau) = b\tau(1-\tau)$. Then $\alpha_1 = 0$, $\alpha_2 = 1$, $a_1 = b$, $a_2 = -b$, and Eq. (12) has the form

$$b\left(\frac{1}{\beta} + \frac{1}{1-\beta}\right) + 1 = 0$$

This implies that

$$\beta_1 = \tfrac{1}{2} + (\tfrac{1}{4} + b)^{1/2}, \qquad \beta_2 = \tfrac{1}{2} - (\tfrac{1}{4} + b)^{1/2}, \qquad \alpha_Q = (\tfrac{1}{4} + b)^{1/2} - \tfrac{5}{2}$$

[$\alpha_Q = 1$ if $b = 12$, in accord with (3.3)]. Hence

$$x_\infty(k) = \frac{1}{a}\eta(k) + \frac{b}{a} k^{-\alpha_Q - 1} \int_k^\infty \eta(\xi)\, \xi^{\alpha_Q}\, d\xi$$

$$-\frac{b}{a} k^{\alpha_Q+4} \int_k^\infty \eta(\xi)\, \xi^{-\alpha_Q-5}\, d\xi$$

Note that some exact solutions of Eq. (8) are presented in the recent work by Derrida and Flyvbjerg.[9]

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